HILBERT SERIES OF RESIDUAL INTERSECTIONS

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Abstract We find explicit formulas for the Hilbert series of residual intersections of a scheme in terms of the Hilbert series of its conormal modules. In a previous paper we proved that such formulas should exist. We give applications to the dimension of secant varieties of surfaces and three-folds.

Introduction

Let $M = \oplus_{i\in \mathbf{Z}} M_i$ be a finitely generated graded module over the homogeneous coordinate ring R of a projective variety over a field k. The *Hilbert series* (sometimes called the Hilbert-Poincaré series) of M, which we will denote $[\![M]\!]$, is the Laurent series

$$[\![M]\!] = \sum (\dim M_i)t^i.$$

If $Z \subset \mathbf{P}^n := \mathbf{P}^n_k$ is a scheme, then the Hilbert series of Z is by definition the Hilbert series of the homogeneous coordinate ring of Z. Of course this Hilbert series contains the data of the Hilbert polynomial of Z as well.

Sometimes interesting geometric data (such as the dimension of a secant variety) can be described in terms of *residual intersections* in the sense of Artin and Nagata [2], and the purpose of this paper is to compute the Hilbert series of such schemes. Here is the definition: let $X \subset Y \subset \mathbf{P}^n$ be closed subschemes of $\mathbf{P}^n$, let R be the homogeneous coordinate ring of Y, and let $I_X \subset R$ be the ideal of X in Y. A scheme $Z \subset Y$ is called an s-residual intersection of X in Y if Z is defined by an ideal of the form $I_Z = (f_1, \ldots, f_s) :_R I_X$, with $f_1, \ldots, f_s$ homogeneous elements in I_X, and Z is of codimension at least s in Y.

We wish to derive formulas for the Hilbert series of Z in terms of information about X and the degrees of the polynomials f_i. In our previous paper [6] we showed that this is sometimes possible in principle: under certain hypotheses the Hilbert series of Z does not vary if we change the polynomials f_i, keeping their degrees fixed. In this paper we make this more precise by giving formulas—under somewhat stronger hypotheses—for the Hilbert series of Z in terms of the degrees of the f_i and the Hilbert series of finitely many modules of the form $\omega_R/I_X^m\omega_R$, where ω_R denotes the canonical module of R.

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For example, suppose that $Y = \mathbf{P^n}$ and X is locally a complete intersection (for instance, smooth). If $f_1 \dots, f_s$ are homogeneous elements of degree d of $I = I_X$ such that $\mathfrak{R} := (f_1, \dots, f_s) : I$ has codimension $\geqslant s$ then the Hilbert series of the homogeneous coordinate ring of the scheme Z defined by $\mathfrak{R}$ differs from that of a complete intersection defined by s forms of degree d by

$$\sum_{j=g}^{s} (-1)^{n+j} \binom{s}{j} t^{jd} [\![\omega_R / I^{j-g+1} \omega_R]\!](t^{-1}) + \text{a polynomial.}$$

The polynomial remainder term is present because we have made assumptions only on the scheme, and not on the homogeneous coordinate ring. Here the expression $[\![R/I^{j-g+1}]\!](t^{-1})$ denotes the Laurent series obtained by writing $[\![R/I^{j-g+1}]\!]$ as a rational function in t, substituting t^{-1} for t, and rewriting the result as a Laurent series. Up to a small shift in notation, this is the formula given in Remark 1.5b (where the case of forms f_i of different degrees is also treated.)

In applications, one sometimes needs to know only whether the s-residual intersection Z actually has codimension exactly s in Y; for example, we will use such information in Section 3 to say when the secant varieties of certain (possibly singular) surfaces and smooth 3-folds have dimension less than the expected dimension. For this purpose it is enough to know just one coefficient of the Hilbert polynomial of Z, that corresponding to the degree of the codimension s component of Z. More generally, we show how to use partial information about X to compute just the first k coefficients of the Hilbert polynomial of Z.

Consider the case where Y is Gorenstein and X is Cohen-Macaulay. Suppose, further, that locally in codimension $i < s$, the subscheme $X \subset Y$ can be defined by i equations, and that, for $j \leqslant s - g$,

$$\operatorname{depth} \mathcal{I}_X^j / \mathcal{I}_X^{j+1} \geqslant \dim X - j.$$

If Z is any s-residual subscheme of X in Y, then the Hilbert polynomial of Z may be written in terms of the Hilbert polynomials of $\mathcal{I}_X^j$ for $j \leqslant s - g + 1$ (the explicit formula is given in Theorem 1.9). Moreover, if X and Y satisfy some of our hypotheses only up to some codimension r, then the formula gives the first r coefficients of the Hilbert polynomial of Z.

Our formulas are derived in Section 1, which is the technical heart of the paper. To prove them we need to adapt the arguments of Ulrich [19]. The delicate point is the use in that paper of the isomorphism $R \simeq \omega_R$ that holds for a Gorenstein ring. Since our rings are Gorenstein only up to a certain codimension r, we know only that ω_R is a line bundle locally in codimension r, and this does not suffice to determine its Hilbert series. Thus the module ω_R must be brought into play. For other work along these lines, see Cumming [7].

In case where Y is Gorenstein, the sheaves $\mathcal{I}_X^j / \mathcal{I}_X^{j+1}$ themselves play the crucial role in our formulas. If X were locally a complete intersection scheme, then $\mathcal{I}_X / \mathcal{I}_X^2$ would be a vector bundle and $\mathcal{I}_X^j / \mathcal{I}_X^{j+1}$ would be its j-th symmetric power, so it is reasonable to hope that for "nice" ideals I the Hilbert series of the first few conormal modules should determine the rest. We prove a general theorem of this kind in Section 2, and carry out the reduction in some particular cases. For instance, if $s = \operatorname{codim}(X)$ then the degree of an s-residual intersection scheme Z in $Y = \mathbb{P}^n$ may be calculated immediately from Bézout's Theorem: $\deg Z = (\prod_i \deg f_i) - \deg X$. This was extended to a formula in the case $s = \operatorname{codim}_Y X + 1$ by Stückrad [18] and to the case $s = \operatorname{codim}_Y X + 2$ by

Huneke and Martin [16]. Our formula gives an answer in general, and we work this out explicitly for the case $s = \operatorname{codim}_Y X + 3$.

In Section 3 we apply our results to the study of secant loci. Our general theorems imply conditions in terms of Chern classes of a smooth embedded three-fold for the degeneracy of the secant locus in terms of Chern classes and in terms of certain Hilbert coefficients. We also recover the analogous criteria for surfaces with mild singularities, a case already by Dale [7a] and others.

1. Formulas for the Hilbert series of residual intersections

To deal with rings R that are not equidimensional, we define the *true codimension* of a prime ideal $\mathfrak{p} \subset R$ to be $\dim R - \dim R/\mathfrak{p}$. We say that an ideal I satisfies the condition $*G_s$ if, for every prime $\mathfrak{p}$ in $V(I)$ of true codimension $< s$, the minimal number of generators of $I_\mathfrak{p}$ is at most $\dim R_\mathfrak{p}$ (the usual condition G_s is the same but for codimension instead of true codimension).

Definition 1.1. Let R be a graded ring and M, N graded R-modules. We say that M and N are *equivalent up to true codimension* r, and write $M \underset{r}{\cong} N$, if there exist graded R-modules $W_1, \ldots, W_n$ with $W_1 = M$, $W_n = N$ and homogeneous maps $W_i \to W_{i+1}$ or $W_{i+1} \to W_i$, for $1 \leqslant i \leqslant n-1$, which are isomorphisms locally in true codimension r. A homogeneous map which is an isomorphism up to true codimension r will be denoted by $\xrightarrow[r]{\sim}$.

Saying that $M \underset{r}{\cong} N$ is of course much stronger than saying that M and N are isomorphic locally at each prime of true codimension $\leqslant r$. For example, any two modules M and N that represent line bundles on a projective variety of dimension r satisfy the latter condition, but $M \underset{r}{\cong} N$ implies that they represent isomorphic line bundles! The need to provide explicit maps that are locally isomorphisms in some true codimension between modules that are not in fact isomorphic is what makes the work in this section delicate.

If R is a Noetherian standard graded algebra over a field and M a finitely generated graded R-module, we will denote the Hilbert series of M by $[\![M]\!]$. If $M \underset{r}{\cong} N$ then $[\![M]\!] - [\![N]\!]$, written as a rational function, has a pole of order less than $\dim R - r$ at 1; we will write this as $[\![M]\!] \underset{r}{\equiv} [\![N]\!]$ and say that these series are r*-equivalent*. Thus if $r = \dim R - 1 =: d$ then $[\![M]\!] \underset{r}{\equiv} [\![N]\!]$ means that the Hilbert polynomials of M and N agree, and in general if $r < \dim R$ then $[\![M]\!] \underset{r}{\equiv} [\![N]\!]$ means that the Hilbert polynomials of M and N, written in the form

$$a_d \binom{d+t}{d} + a_{d-1} \binom{d-1+t}{t} + \cdots,$$

have the same coefficients of a_s for $s \geqslant \dim R - 1 - r$.

We will also extend the notation $\underset{r}{\equiv}$ to arbitrary series that are rational functions with no pole outside 1, by the same requirement, as soon as $\dim R$ is clear from the context.

The substitution $t \mapsto t^{-1}$ is a well-defined automorphism of the ring $\mathbf{Z}[t, t^{-1}, (1-t)^{-1}]$, since $(1-t)^{-1} = -t(1-t)^{-1} \in \mathbf{Z}[t, t^{-1}, (1-t)^{-1}]$.

Lemma 1.2. Let R be a positively graded Noetherian algebra over a field k. If for each prime $\mathfrak{p}$ of dimension $\geqslant \dim R - r$, the ring $R_\mathfrak{p}$ is Cohen-Macaulay of dimension $\dim R - \dim R/\mathfrak{p}$, then

$$[\![\omega_R]\!](t) \underset{r}{\equiv} (-1)^{\dim R} [\![R]\!](t^{-1}).$$

Proof. Dualize a free resolution of R over a polynomial ring S, and note that all the homology modules other than $\operatorname{Ext}_S^{\operatorname{codim}_S(R)}(R,\omega_S)$ are supported in codimension $> r$. $\square$

We next adapt some results of [6] and [20] to our context.

Suppose that R is a local Cohen-Macaulay ring and I is an ideal of height g. In the rest of this section we will often use the condition that $\operatorname{depth} R/I^j \geqslant \dim R/I - j + 1$ for $1 \leqslant j \leqslant s-g$. These conditions are satisfied if I satisfies G_s and if moreover, I has the sliding depth property or, more restrictively, is strongly Cohen-Macaulay (which means that for every i, the i-th Koszul homology H_i of a generating set $h_1,\ldots,h_n$ of I satisfies depth $H_i \geqslant \dim R - n + i$ or is Cohen-Macaulay, respectively) ([13, 3.3] and [15, 3.1]). The latter condition always holds if I is a Cohen-Macaulay almost complete intersection or a Cohen-Macaulay deviation 2 ideal of a Gorenstein ring [3]. It is also satisfied for any ideal in the linkage class of a complete intersection [14, 1.11]. Standard examples include perfect ideals of grade 2 ([1] and [9]) and perfect Gorenstein ideals of grade 3 [21].

Lemma 1.3. *Let R be a finitely generated positively graded algebra over a factor ring of a local Gorenstein ring and write $\omega = \omega_R$. Let I be a homogeneous ideal of height g, let $f_1,\ldots,f_s$ be forms contained in I of degrees $d_1,\ldots,d_s$, write $\mathfrak{A}_i := (f_1,\ldots,f_i)$, $\mathfrak{A} := \mathfrak{A}_s$, and $\mathfrak{R}_i = \mathfrak{A}_i : I$. Assume that $\operatorname{ht}\mathfrak{R}_i \geqslant i$ for $1 \leqslant i \leqslant s$ and $\operatorname{ht} I + \mathfrak{R}_i \geqslant i+1$ for $1 \leqslant i \leqslant s-1$. Further suppose that, locally off $V(\mathfrak{A})$, the elements $f_1,\ldots,f_s$ form a weak regular sequence on R and on ω, and that locally in true codimension r in R along $V(\mathfrak{A})$, the ring R is Gorenstein and* $\operatorname{depth} R/I^j \geqslant \dim R/I - j + 1$ *for $1 \leqslant j \leqslant s-g$. Then :*

(a) $(R/\mathfrak{R}_{i-1})(-d_i) \underset{r}{\xrightarrow{\sim}} \mathfrak{A}_i/\mathfrak{A}_{i-1}$ *via multiplication by f_i for $1 \leqslant i \leqslant s$.*

(b) $0 \to (\omega I^j/\omega\mathfrak{A}_{i-1}I^{j-1})(-d_i) \xrightarrow{\cdot f_i} \omega I^{j+1}/\omega\mathfrak{A}_{i-1}I^j \longrightarrow \omega I^{j+1}/\omega\mathfrak{A}_i I^j \to 0$ *is a complex that is exact locally in true codimension r for $1 \leqslant i \leqslant s$ and $\min\{1, i-g\} \leqslant j \leqslant s-g$.*

(c) $\operatorname{Ext}_R^i(R/\mathfrak{R}_i,\omega) \underset{r}{\cong} (\omega I^{i-g+1}/\omega\mathfrak{A}_i I^{i-g})(d_1+\cdots+d_i)$ *for $0 \leqslant i \leqslant s$, if locally in true codimension r in R along $V(\mathfrak{A})$,* $\operatorname{depth} R/I^j \geqslant \dim R/I - j + 1$ *for $1 \leqslant j \leqslant s-g+1$.*

Proof. Adjoining a variable to R and to I and localizing, we may suppose that $\operatorname{grade} I > 0$. Furthermore, our assumptions imply that I is G_s and satisflies the Artin-Nagata condition AN_{s-1} locally in true codimension r; see [20, 2.9(a)]. Likewise, in the setting of (c), AN_s holds locally in true codimension r.

Part (a) holds along $V(\mathfrak{A})$ by [20, 1.7(g)], and off $V(\mathfrak{A})$ because $f_1,\ldots,f_i$ form a weak regular sequence. Moreover, the sequence of (b) is obviously a complex and it is exact in true codimension r by [20, 2.7(a)].

For the proof of (c), we induct on i. For $i=0$, our assertion is clear since $\operatorname{grade} I > 0$ and therefore $R/\mathfrak{R}_0 = R$. Assuming that the assertion holds for $R_i = R/\mathfrak{R}_i$ for some i, $0 \leqslant i \leqslant s-1$, we are going to prove our claim for $R_{i+1} = R/\mathfrak{R}_{i+1}$. To this end we may suppose $r \geqslant i+1$.

We first wish to prove that

$$\operatorname{Ext}_R^{i+1}(R_i/(f_{i+1}R_i : IR_i),\omega) \underset{r}{\cong} [I \operatorname{Ext}_R^i(R_i,\omega)/f_{i+1}\operatorname{Ext}_R^i(R_i,\omega)](d_{i+1}). \tag{1}$$

Using the exact sequence

$$0 \to R_i/(0 :_{R_i} f_{i+1}R_i)(-d_{i+1}) \xrightarrow{\cdot f_{i+1}} R_i \longrightarrow R_i/f_{i+1}R_i \to 0$$

we obtain a long exact sequence

$$\cdots \operatorname{Ext}^i_R(R_i,\omega) \xrightarrow{\cdot f_{i+1}} \operatorname{Ext}^i_R(R_i/(0:_{R_i} f_{i+1}R_i),\omega)(d_{i+1}) \to \operatorname{Ext}^{i+1}_R(R_i/f_{i+1}R_i,\omega) \to \operatorname{Ext}^{i+1}_R(R_i,\omega)\cdots$$

Since the support in R of $(0:_{R_i} f_{i+1})$ has true codimension $\geqslant r+1 > i$ by part (a), we have $\operatorname{Ext}^i_R(R_i/(0:_{R_i} f_{i+1}R_i),\omega) \xrightarrow[r]{\sim} \operatorname{Ext}^i_R(R_i,\omega)$ via the natural map. Furthermore $\operatorname{Ext}^{i+1}_R(R_i,\omega) \underset{r}{\cong} 0$; as locally up to true codimension r on $V(\mathfrak{A})$, R_i is Cohen-Macaulay of true codimension i by [20, 1.7(a)], whereas locally up to true codimension r off $V(\mathfrak{A})$, R_i is defined by the weak regular sequence $f_1,\dots,f_i$ and hence has projective dimension at most i. Therefore

$$\left[\operatorname{Ext}^i_R(R_i,\omega)/f_{i+1}\operatorname{Ext}^i_R(R_i,\omega)\right](d_{i+1}) \underset{r}{\cong} E := \operatorname{Ext}^{i+1}_R(R_i/f_{i+1}R_i,\omega).$$

Hence, to prove (1), it suffices to show that

$$I\ \operatorname{Ext}^{i+1}_R(R_i/f_{i+1}R_i,\omega) \underset{r}{\cong} \operatorname{Ext}^{i+1}_R(R_i/(f_{i+1}R_i : IR_i),\omega).$$

The map $R_i/f_{i+1}R_i \to R_i/(f_{i+1}R_i :_{R_i} IR_i)$ induces a map

$$\phi:\ \operatorname{Ext}^{i+1}_R(R_i/(f_{i+1}R_i : IR_i),\omega) \to \operatorname{Ext}^{i+1}_R(R_i/f_{i+1}R_i,\omega).$$

We prove that locally in true codimension r in R, the map ϕ is injective, and its image coincides with $I\,E$, which gives

$$\operatorname{im}\phi \xrightarrow[r]{\sim} \operatorname{im}\phi + I\,E \xleftarrow[r]{\sim} I\,E.$$

This is trivial locally off $V(\mathfrak{A})$ because on this locus $I=R$ and $f_{i+1}R_i : IR_i = f_{i+1}R_i$. Therefore, we may localize to assume that R is a local ring of dimension at most r and $\mathfrak{A} \neq R$. Of course we may suppose $R_i \neq 0$. In this case R is Gorenstein, R_i is Cohen-Macaulay of codimension i, and f_{i+1} is a non zerodivisor on R_i by [20, 1.7(f)]. Let $S = R_i/f_{i+1}R_i$.

The natural equivalence of functors $\operatorname{Ext}^{i+1}_R(—,\omega) \simeq \operatorname{Hom}_S(—,\omega_S)$ together with the exact sequence

$$0 \to 0:_S IS \longrightarrow S \longrightarrow S/0:_S IS \longrightarrow 0$$

yield a commutative diagram with an exact row

$$\begin{tikzcd}
 & \operatorname{Ext}^{i+1}_R(R_i/(f_{i+1}R_i : IR_i),\omega) \arrow[r, "\phi"] \arrow[d, "\simeq"] & E \arrow[d, "\simeq"] & & \\
0 \arrow[r] & \operatorname{Hom}_S(S/0:_S IS,\omega_S) \arrow[r, "\psi"] & \omega_S \arrow[r] & \operatorname{Hom}_S(0:_S IS,\omega_S) \arrow[r] & 0
\end{tikzcd}$$

The last map is surjective because $S/(0:_S IS) = R_{i+1}$ by [20,1.7(f)] and R_{i+1} is a maximal Cohen-Macaulay S-module. Now ϕ is injective and the desired equality $\operatorname{im}\phi = I\,E$ follows once we have shown that $\operatorname{im}\psi = I\omega_S$. For this it suffices to prove

(2) $$\operatorname{coker}\psi \simeq \omega_S/I\omega_S;$$

for then $I\omega_S \subset \text{im } \psi$ and we have the natural epimorphism of isomorphic modules $\omega_S/I\omega_S \to \text{coker } \psi$, which is necessarily an isomorphism. We first argue that $\omega_S/I\omega_S$ is a maximal Cohen-Macaulay S-module. Indeed, [20, 2.7(c)] gives $\mathfrak{R}_i \cap I^{i-g+2} = \mathfrak{A}_i I^{i-g+1}$, which implies

$$\mathfrak{A}_i I^{i-g} \cap I^{i-g+2} = \mathfrak{A}_i I^{i-g+1}. \tag{3}$$

Hence by our induction hypothesis,

$$I\omega_S \simeq I^{i-g+2}/(\mathfrak{A}_i I^{i-g} \cap I^{i-g+2} + f_{i+1} I^{i-g+1}) = I^{i-g+2}/J_{i+1} I^{i-g+1}.$$

But the latter is indeed a maximal Cohen-Macaulay S-module according [20, 2.7(b)]. Thus

$$\omega_S/I\omega_S \simeq \text{Hom}_S(\text{Hom}_S(\omega_S/I\omega_S, \omega_S), \omega_S) \simeq \text{Hom}_S(\text{Hom}_S(S/IS, S), \omega_S) \simeq \text{Hom}_S(0 :_S IS, \omega_S).$$

This completes the proof of (2), and hence of (1).

Now

$$I\text{Ext}^i_R(R_i, \omega)/f_{i+1}\text{Ext}^i_R(R_i, \omega) \underset{r}{\cong} (\omega I^{i-g+2}/(\omega\mathfrak{A}_i I^{i-g} \cap \omega I^{i-g+2} + \omega f_{i+1} I^{i-g+1}))(d_1 + \cdots + d_i)$$

by our induction hypothesis, and using (3) one sees that

$$I\text{Ext}^i_R(R_i, \omega)/f_{i+1}\text{Ext}^i_R(R_i, \omega) \underset{r}{\cong} (\omega I^{i-g+2}/\omega\mathfrak{A}_{i+1} I^{i-g+1})(d_1 + \cdots + d_i). \tag{4}$$

On the other hand, $R_{i+1} \underset{r}{\xrightarrow{\sim}} R_i/(f_{i+1}R_i : IR_i)$ according to [20, 1.7(f)], and hence

$$\text{Ext}^{i+1}_R(R_{i+1}, \omega) \underset{r}{\cong} \text{Ext}^{i+1}_R(R_i/(f_{i+1}R_i : IR_i), \omega). \tag{5}$$

Now combining (5), (1), (4) concludes the proof of part (c). □

We write $\sigma_m(t_1, \ldots, t_s)$ for the m-th elementary symmetric function.

Theorem 1.4. Let R be a standard graded Noetherian algebra over a field. Write $n = \dim R$ and $\omega = \omega_R$, and let I be a homogeneous ideal of height g satisfying $*G_s$. Let $f_1, \ldots, f_s$ be forms contained in I of degrees $d_1, \ldots, d_s$, write $\Delta_s := \prod_{i=1}^s (1 - t^{d_i})$, $\mathfrak{A} = (f_1, \ldots, f_s)$, $\mathfrak{R} = \mathfrak{A} : I$, and assume that $\text{ht}\, \mathfrak{R} \geqslant s$. For each prime of true codimension $\leqslant r$ suppose:

- If $\mathfrak{p} \notin V(\mathfrak{A})$, then the elements $f_1, \ldots, f_s$ form a weak regular sequence on $R_\mathfrak{p}$ and on $\omega_\mathfrak{p}$.
- If $\mathfrak{p} \in V(\mathfrak{A})$, then the ring $R_\mathfrak{p}$ is Gorenstein of dimension equal to the true codimension of $\mathfrak{p}$ and $\text{depth}\, R_\mathfrak{p}/I^j_\mathfrak{p} \geqslant \dim R_\mathfrak{p}/I_\mathfrak{p} - j + 1$ for $1 \leqslant j \leqslant s - g$.

(a)

$$[\![R/\mathfrak{A}]\!](t) \underset{r}{\equiv} \Delta_s [\![R]\!](t) - (-1)^{n-g} \sum_{j=1}^{s-g} (-1)^j \sigma_{g+j}(t^{d_1}, \ldots, t^{d_s}) [\![\omega/I^j\omega]\!](t^{-1}).$$

(b) If furthermore, locally in true codimension r in R along $V(\mathfrak{A})$, depth $R/I^{s-g+1} \geqslant \dim R/I - s + g$, then

$$[\![R/\mathfrak{R}]\!](t) \underset{r}{\equiv} \Delta_s [\![R]\!](t) - (-1)^{n-g} \sum_{j=1}^{s-g+1} (-1)^{j-1} \sigma_{g+j-1}(t^{d_1}, \ldots, t^{d_s}) [\![\omega/I^j\omega]\!](t^{-1}).$$

Proof. For $0 \leqslant i \leqslant s$, write $\mathfrak{A}_i = (f_1, \ldots, f_i)$, $\mathfrak{R}_i = \mathfrak{A}_i : I$. Applying repeatedly a general position argument (see [6, 2.5]), we may assume that $\operatorname{ht} \mathfrak{R}_i \geqslant i$ and $\operatorname{ht} I + \mathfrak{R}_i \geqslant i+1$ for $0 \leqslant i \leqslant s-1$.

We first notice that for $0 \leqslant i \leqslant s$ and $i - g \leqslant j \leqslant s - g$,

$$[\![\omega \mathfrak{A}_i I^j]\!] \underset{r}{\equiv} \sum_{\ell=1}^{i} (-1)^{\ell+1} \sigma_\ell(t^{d_1}, \ldots, t^{d_i}) [\![\omega I^{j-\ell+1}]\!],$$

which can be easily deduced from Lemma 1.3(b) using induction on i.

Now let $0 \leqslant i \leqslant s-1$ for (a), or $0 \leqslant i \leqslant s$ for (b), respectively. Then by Lemma 1.3(c),

$$\begin{aligned} [\![\operatorname{Ext}^i_R(R/\mathfrak{R}_i, \omega)]\!](t) &\underset{r}{\equiv} t^{-(d_1+\cdots+d_i)} ([\![\omega I^{i-g+1}]\!] - [\![\omega \mathfrak{A}_i I^{i-g}]\!])(t) \\ &\underset{r}{\equiv} t^{-(d_1+\cdots+d_i)} \sum_{\ell=0}^{i} (-1)^\ell \sigma_\ell(t^{d_1}, \ldots, t^{d_i}) [\![\omega I^{i-g-\ell+1}]\!](t). \end{aligned} \tag{1}$$

Locally in true codimension r in R, $R/\mathfrak{R}_i$ is either zero or Cohen-Macaulay of codimension i by [20, 2.9 and 1.7(a)]. Therefore

$$[\![\operatorname{Ext}^i_R(R/\mathfrak{R}_i, \omega)]\!](t) \underset{r}{\equiv} (-1)^{n-i} [\![R/\mathfrak{R}_i]\!](t^{-1}),$$

as can be easily seen by dualizing a homogeneous finite free resolution of $R/\mathfrak{R}_i$ over a polynomial ring and using Lemma 1.2. Now by Lemma 1.3(c) and (1),

$$\begin{aligned} [\![R/\mathfrak{R}_i]\!](t) &\underset{r}{\equiv} (-1)^{n-i} [\![\operatorname{Ext}^i_R(R/\mathfrak{R}_i, \omega)]\!](t^{-1}) \\ &\underset{r}{\equiv} (-1)^{n-i} t^{d_1+\cdots+d_i} \sum_{\ell=0}^{i} (-1)^\ell \sigma_\ell(t^{-d_1}, \ldots, t^{-d_i}) [\![\omega I^{i-g-\ell+1}]\!](t^{-1}) \\ &= (-1)^{n-i} \sum_{\ell=0}^{i} (-1)^\ell \sigma_{i-\ell}(t^{d_1}, \ldots, t^{d_i}) [\![\omega I^{i-g-\ell+1}]\!](t^{-1}). \end{aligned}$$

Therefore

$$[\![R/\mathfrak{R}_i]\!](t) \underset{r}{\equiv} (-1)^{n-g} \sum_{j=-g+1}^{i-g+1} (-1)^{j+1} \sigma_{g+j-1}(t^{d_1}, \ldots, t^{d_i}) [\![\omega I^j]\!](t^{-1}). \tag{2}$$

Now part (b) follows since

$$\begin{aligned} \Delta_i [\![R]\!](t) \underset{r}{\equiv} \Delta_i (-1)^n [\![\omega]\!](t^{-1}) &= (-1)^{n-i} \sum_{\ell=0}^{i} (-1)^\ell \sigma_{i-\ell}(t^{d_1}, \ldots, t^{d_i}) [\![\omega]\!](t^{-1}) \\ &= (-1)^{n-g} \sum_{j=-g}^{i-g} (-1)^j \sigma_{g+j}(t^{d_1}, \ldots, t^{d_i}) [\![\omega]\!](t^{-1}). \end{aligned}$$

To see (a) notice that by Lemma 1.3(a),

$$[\![R/\mathfrak{A}_i]\!](t) = [\![R/\mathfrak{A}_{i-1}]\!](t) - t^{d_i}[\![R/\mathfrak{R}_{i-1}]\!](t)$$

for $1 \leqslant i \leqslant s$. Now by induction on i using (2), one shows that

$$[\![R/\mathfrak{A}_i]\!](t) \underset{r}{\equiv} (-1)^{n-g} \sum_{j=-g}^{i-g} (-1)^j \sigma_{g+j}(t^{d_1}, \ldots, t^{d_i})[\![\omega I^j]\!](t^{-1}),$$

from which (a) can be easily deduced. $\square$

Remark 1.5. If in Theorem 1.4, R is a polynomial ring in n variables, then the formulas of that theorem take the following form:

(a) $$[\![R/\mathfrak{A}]\!](t) \underset{r}{\equiv} \Delta_s[\![R]\!](t) - (-t)^{-n} \sum_{j=1}^{s-g} (-1)^{g+j} \sigma_{g+j}(t)[\![R/I^j]\!](t^{-1});$$

(b) $$[\![R/\mathfrak{R}]\!](t) \underset{r}{\equiv} \Delta_s[\![R]\!](t) - (-t)^{-n} \sum_{j=1}^{s-g+1} (-1)^{g+j-1} \sigma_{g+j-1}(t)[\![R/I^j]\!](t^{-1}).$$

Lemma 1.6. Write $\Delta_s(t) = (1-t)^s \sum_{k \geqslant 0} c_k(d_1, \ldots, d_s)(1-t)^k$. Then

$$c_k(d_1, \ldots, d_s) = (-1)^k \sum_{\substack{i_1 \geqslant 1, \ldots, i_s \geqslant 1 \\ i_1 + \cdots + i_s = k+s}} \prod_{j=1}^{s} \binom{d_j}{i_j}.$$

Proof. Write $P_j(t) = \sum_{\ell=0}^{d_j-1} t^\ell$ and notice that $\Delta_s(t) = (1-t)^s \prod_{j=1}^s P_j(t)$. Now $P_j^{(m)}(1) = \sum_{\ell=0}^{d_j-1} m!\binom{\ell}{m} = m!\binom{d_j}{m+1}$. Hence

$$\begin{aligned}\Big(\prod_{j=1}^{s} P_j\Big)^{(k)}(1) &= \sum_{\substack{m_1 \geqslant 0, \ldots, m_s \geqslant 0 \\ m_1 + \cdots + m_s = k}} \frac{k!}{m_1! \cdots m_s!} \prod_{j=1}^{s} P_j^{(m_j)}(1) \\ &= k! \sum_{\substack{m_1 \geqslant 0, \ldots, m_s \geqslant 0 \\ m_1 + \cdots + m_s = k}} \prod_{j=1}^{s} \binom{d_j}{m_j + 1}.\end{aligned}$$

This yields our formula since $c_k(d_1, \ldots, d_s) = \frac{(-1)^k}{k!}\Big(\prod_{j=1}^s P_j\Big)^{(k)}(1)$. $\square$

Lemma 1.7. Let P be a numerical polynomial written in the form $P(t) = \sum_{i=0}^m (-1)^i e_i \binom{t+m-i}{m-i}$. For an integer d define the polynomial $Q(t) = P(-t+d)$, and write $Q(t) = \sum_{i=0}^m (-1)^i h_i \binom{t+m-i}{m-i}$. Then

$$h_i = (-1)^m \sum_{k=0}^{i} (-1)^k \binom{d+m+1-k}{i-k} e_k.$$

Proof. We first notice that for integers r and $n \geqslant 0$, one has the following identities of numerical polynomials :

$$\binom{-t+n}{n} = (-1)^n \binom{t-1}{n}, \tag{3}$$

$$\binom{t+r+n}{n} = \sum_{\ell=0}^{n} \binom{r-1+\ell}{\ell}\binom{t+n-\ell}{n-\ell}, \tag{4}$$

where the first is obvious and the second can be easily proved by induction on n.

Now

$$\begin{aligned}
Q(t) &= \sum_{k=0}^{m}(-1)^k e_k \binom{-t+d+m-k}{m-k} \\
&= (-1)^m \sum_{k=0}^{m} e_k \binom{t-d-1}{m-k} \quad \text{by (3)} \\
&= (-1)^m \sum_{k=0}^{m} e_k \binom{t+(-d-1-m+k)+(m-k)}{m-k} \\
&= (-1)^m \sum_{k=0}^{m} e_k \sum_{\ell=0}^{m-k} \binom{-d-1-m+k-1+\ell}{\ell}\binom{t+m-k-\ell}{m-k-\ell} \quad \text{by (4)} \\
&= (-1)^m \sum_{i=0}^{m} \left(\sum_{k=0}^{i} \binom{-d-m-2+i}{i-k} e_k \right) \binom{t+m-i}{m-i},
\end{aligned}$$

where $\binom{-d-m-2+i}{i-k} = (-1)^{i+k}\binom{d+m+1-k}{i-k}$ by (3). □

Recall that the Hilbert series $[\![M]\!]$ of a finitely generated graded module M over a homogeneous ring over a field is element of the ring $\mathbf{Z}[t, t^{-1}, (1-t)^{-1}] \subset \mathbf{Z}[\![t]\!][t^{-1}]$. In general, any $S \in \mathbf{Z}[t, t^{-1}, (1-t)^{-1}]$ can be written uniquely in the form

$$S(t) = \sum_{i=0}^{D-1} (-1)^i e_i \frac{1}{(1-t)^{D-i}} + F$$

where $e_i \in \mathbf{Z}$ and $F \in \mathbf{Z}[t, t^{-1}]$. The coefficients e_i can be computed as $e_i(M) = \frac{\partial^i P}{i!}(1)$, where $P(t) = S(t)(1-t)^D$. We call

$$Q(t) = \sum_{i=0}^{D-1} (-1)^i e_i \binom{t+D-1-i}{D-1-i} \in \mathbf{Q}[t]$$

the polynomial associated to S. Its significance is that if we write $S = \sum_{i \in \mathbf{Z}} c_i t^i$, then $c_i = Q(i)$ for $i \gg 0$.

Remark 1.8. In the case where $S(t) = [\![M]\!](t)$, we can take D to be any integer $\geqslant \dim M$, and we define $e_i^D(M) := e_i$. If $D = \dim M$, we simply set $e_i(M) := e_i^D(M)$. Notice that $e_0(M)$ is the multiplicity (or degree) of M. The polynomial associated to $[\![M]\!](t)$ is the Hilbert polynomial of M, which we denote by $[M](t)$.

Theorem 1.9. *Write* $e_\ell(d_1, \ldots, d_s) = \sum_{\substack{i_1 \geqslant 1, \ldots, i_s \geqslant 1 \\ i_1 + \cdots + i_s = \ell + s}} \prod_{j=1}^s \binom{d_j}{i_j}$.

(a) *With the assumptions of Theorem 1.4(a),*

$$\begin{aligned} e_i^{n-s}(I/\mathfrak{A}) = {} & \sum_{k=0}^{i} e_{i-k}(d_1, \ldots, d_s) e_k(R) - (-1)^{s-g} e_{s-g+i}(R/I) \\ & - (-1)^{s-g} \sum_{j=1}^{s-g} \sum_{k=0}^{s-g+i} (-1)^{j+k} \sum_{1 \leqslant i_1 < \cdots < i_{g+j} \leqslant s} \binom{d_{i_1} + \cdots + d_{i_{g+j}} + n - g - k}{i + s - g - k} e_k(\omega/I^j\omega) \end{aligned}$$

for $0 \leqslant i \leqslant r - s$.

(b) *With the assumptions of Theorem 1.4(b),*

$$\begin{aligned} e_i^{n-s}(R/\mathfrak{R}) = {} & \sum_{k=0}^{i} e_{i-k}(d_1, \ldots, d_s) e_k(R) \\ & + (-1)^{s-g} \sum_{j=1}^{s-g+1} \sum_{k=0}^{s-g+i} (-1)^{j+k} \sum_{1 \leqslant i_1 < \cdots < i_{g+j-1} \leqslant s} \binom{d_{i_1} + \cdots + d_{i_{g+j-1}} + n - g - k}{i + s - g - k} e_k(\omega/I^j\omega) \end{aligned}$$

for $0 \leqslant i \leqslant r - s$.

Proof. We only prove part (a). First write

$$\begin{aligned} (-1)^{n-g} \sum_{j=1}^{s-g} (-1)^j \sum_{1 \leqslant i_1 < \cdots < i_{g+j} \leqslant s} [\omega/I^j\omega](-t + d_{i_1} + \cdots + d_{i_{g+j}}) &= \sum_{\ell=0}^{n-g-1} (-1)^\ell h_\ell \binom{t + n - g - 1 - \ell}{n - g - 1 - \ell} \\ &= \sum_{\ell=0}^{n-s-1} (-1)^\ell h_\ell^* \binom{t + n - s - 1 - \ell}{n - s - 1 - \ell} \end{aligned}$$

and notice that $h_i^* = (-1)^{s-g} h_{i+s-g}$. Lemma 1.7 gives

$$h_{i+s-g} = -\sum_{j=1}^{s-g} (-1)^j \sum_{1 \leqslant i_1 < \cdots < i_{g+j} \leqslant s} \sum_{k=0}^{i+s-g} \binom{d_{i_1} + \cdots + d_{i_{g+j}} + n - g - k}{i + s - g - k} (-1)^k e_k(\omega/I^j\omega).$$

Now our assertion follows from Theorem 1.4(a) together with Lemma 1.6.

The proof of part (b) is similar, using Theorem 1.4(b) in place of Theorem 1.4(a). □

Remark 1.10. If in Theorem 1.9, R is a polynomial ring in n variables, then the formula in that theorem takes the following form:

$$e_i^{n-s}(I/\mathfrak{A}) = e_i(d_1,\ldots,d_s) - (-1)^{s-g}e_{s-g+i}(R/I)$$
$$- (-1)^{s-g}\sum_{j=1}^{s-g}\sum_{k=0}^{s-g+i}(-1)^{j+k}\sum_{1\leqslant i_1<\cdots<i_{g+j}\leqslant s}\binom{d_{i_1}+\cdots+d_{i_{g+j}}-g-k}{i+s-g-k} e_k(R/I^j)$$

for $0 \leqslant i \leqslant r-s$;

$$e_i^{n-s}(R/\mathfrak{R}) = e_i(d_1,\ldots,d_s)$$
$$+ (-1)^{s-g}\sum_{j=1}^{s-g+1}\sum_{k=0}^{s-g+i}(-1)^{j+k}\sum_{1\leqslant i_1<\cdots<i_{g+j-1}\leqslant s}\binom{d_{i_1}+\cdots+d_{i_{g+j-1}}-g-k}{i+s-g-k} e_k(R/I^j)$$

for $0 \leqslant i \leqslant r-s$.

Proof. Notice that $\omega/I^j\omega \simeq R/I^j(-n)$ and proceed as in the proof of Theorem 1.9. □

Corollary 1.11. Let R be a homogeneous ring over a field, write $n = \dim R$, $\omega = \omega_R$, and assume that R is Gorenstein locally in true codimension $r \geqslant s$. Let I be a homogeneous ideal of height g satisfying G_s, let $f_1,\ldots,f_s$ be forms contained in I of degrees $d_1,\ldots,d_s$, write $\mathfrak{A} = (f_1,\ldots,f_s)$, $\mathfrak{R} = \mathfrak{A} : I$, and assume that locally in true codimension r, $\operatorname{depth} R/I^j \geqslant \dim R/I - j + 1$ for $1 \leqslant j \leqslant s-g$.

Then $\operatorname{ht}\mathfrak{R} \geqslant r+1$ if and only if $\operatorname{ht}\mathfrak{R} \geqslant s$ and

$$(-1)^{s-g}e_0(R)\prod_{j=1}^{s} d_j = e_{s-g}(R/I)$$
$$+\sum_{j=1}^{s-g}\sum_{k=0}^{s-g}(-1)^{j+k}\sum_{1\leqslant i_1<\cdots<i_{g+j}\leqslant s}\binom{d_{i_1}+\cdots+d_{i_{g+j}}+n-g-k}{s-g-k}e_k(\omega/I^j\omega).$$

Proof. One uses Theorem 1.9(a) and [20, 1.7(a)]. □

2. Hilbert series of powers of ideals and degrees of residual intersections.

2.1 Computing Hilbert series of powers.

Motivated by Remark 1.5, showing the usefulness of the Hilbert series of the powers of an ideal, we will focus here on the following question: to what extent does the Hilbert series of the first powers of an ideal determine the Hilbert series of the next powers?

The following Lemma tells us a useful property of a general set of elements of an ideal.

Lemma 2.1. *Let R be a standard graded Cohen-Macaulay ring over an infinite field k, let I a homogeneous ideal, generated by forms of degrees at most d, and let r be an integer with $0 \leqslant r \leqslant \dim R$. Further assume that I satisfies G_{r+1} and has sliding depth locally in codimension r.*

Given $d_i \geqslant d$ for $1 \leqslant i \leqslant r+1]$, there exists a Zariski dense open subset Ω of the affine k-space $I_{d_1}\times\cdots\times I_{d_{r+1}} \simeq \mathbf{A}_k^N$ (where $N = \sum_{i=1}^{r+1}\dim_k I_{d_i}$) such that if $(f_1,\ldots,f_{r+1}) \in \Omega$:

(a) *The ideal* $(f_1, \ldots, f_{r+1})$ *coincides with* I *locally up to codimension* r.

(b) $f_1, \ldots, f_{r+1}$ *is a d-sequence locally up to codimension* r.

Proof. For part (a) we refer to [2, 1.6 (a)] and [10, 3.9], while (b) follows from [6, 3.6(b)]. $\square$

We now choose some degrees (e.g. $d_i = d$ for all i) and polynomials f_i as above. Part two of the lemma implies that the approximation complexes corresponding to $f_1, \ldots, f_{r+1}$, and therefore their different graded components,

$$\mathcal{M}_p : \quad 0 \to H_{r+1} \otimes S_{p-r-1} \to H_r \otimes S_{p-r} \to \cdots \to H_0 \otimes S_p \to 0$$

have homology in positive degrees supported in codimension at least $r+1$. Moreover, $H_0(\mathcal{M}_p)$ coincides with I^p/I^{p+1} locally up to codimension r. See [12] for all of these facts.

Recall that in this sequence, H_q stands for the q-th homology module of the Koszul complex $\mathbf{K}(f_1, \ldots, f_{r+1}; R)$ and S_q is the free R-module generated by monomials of degree q in $r+1$ variables. The maps are homogeneous of degree 0 in the graded case, with the usual weights on the Koszul complex and similarely the weight of $s_{i_1} \cdots s_{i_q} \in S_q$ is $\deg(f_{i_1}) + \cdots + \deg(f_{i_q})$.

Therfore we have the following equalities :

$$(1)_p \qquad [\![I^p/I^{p+1}]\!](t) \underset{r}{\equiv} \sum_{i=0}^{p} (-1)^i s_{p-i}(t^{d_1}, \ldots, t^{d_{r+1}}) [\![H_i]\!](t)$$

for every $p \geqslant 0$, where s_j stands for the sum of all the monomials of degree j in $r+1$ variables (complete symmetric functions). These express the Hilbert series of the modules $H_0, \ldots, H_p$ in terms of the ones of $R/I, \ldots, I^p/I^{p+1}$, and vice versa.

If I has height g, then $H_q = 0$ for $q > r+1-g$. We therefore immediately see that the Hilbert series of all the modules I^p/I^{p+1} are determined, up to r-equivalence, by the knowledge of the Hilbert series of I^p/I^{p+1} for $0 \leqslant p \leqslant r+1-g$, up to r-equivalence.

We now assume that, in addition, R is Gorenstein with a-invariant $a := a(R)$ and is strongly Cohen-Macaulay locally in codimension r. In this case we can use the self-duality of the homology of the Koszul complex to see that only half of the information about the Koszul homology is needed. Indeed, the structure of graded alternating algebra on the homology of the Koszul complex gives a graded map of degree 0,

$$H_p \longrightarrow \mathrm{Hom}_{R/I}(H_{r+1-g-p}, H_{r+1-g}),$$

which is an isomorphism up to codimension r by a theorem of Herzog (see [11, 2.4.1]).

Now we have a collection of graded maps of degree 0 that connect the following modules,

$$\begin{aligned} \mathrm{Hom}_{R/I}(H_{r+1-g-p}, H_{r+1-g}) &\underset{r}{\cong} \mathrm{Hom}_{R/I}(H_{r+1-g-p}, \mathrm{Ext}^g_R(R/I, R)[-(d_1 + \cdots + d_{r+1})]) \\ &\underset{r}{\cong} \mathrm{Hom}_{R/I}(H_{r+1-g-p}, \omega_{R/I}[-a - (d_1 + \cdots + d_{r+1})]) \\ &\underset{r}{\cong} \omega_{H_{r+1-g-p}}[a + (d_1 + \cdots + d_{r+1})]. \end{aligned}$$

From the Cohen-Macaulayness of the modules H_p, locally in codimension at most r, we therefore have:

$$(2)_p \qquad [\![H_p]\!](t) \underset{r}{\equiv} t^{a+(d_1+\cdots+d_{r+1})} (-1)^{\dim R/I} [\![H_{r+1-g-p}]\!](t^{-1}).$$

Moreover, the Euler characteristic of the homology of the Koszul complex depends only upon the degrees $d_1, \ldots, d_{r+1}$, namely

$$(3) \qquad \sum_{p=0}^{r+1-g} (-1)^p [\![H_p]\!] = \Delta_{r+1}[\![R]\!] \underset{r}{\equiv} 0.$$

We now have put together all the formulas needed to effectively compute what we state in the next theorem.

Remark that the result of the computation does not depend on the choice of the d_i's. Thus we may choose $d_i = 0$ for all i, the intermediate steps have no meaning (e.g. $[\![H_p]\!]$ may not have positive coefficients), but the information that we extract from the computation is the same. Out of this remark, one may use the following :

$$(1)_p \qquad [\![I^p/I^{p+1}]\!](t) \underset{r}{\equiv} \sum_{i=0}^{p} (-1)^i \binom{r+p-i}{r} [\![H_i]\!](t),$$

$$(2)_p \qquad [\![H_p]\!](t) \underset{r}{\equiv} t^a (-1)^{\dim R/I} [\![H_{r+1-g-p}]\!](t^{-1}),$$

$$(3) \qquad \sum_{p=0}^{r+1-g} (-1)^p [\![H_p]\!] \underset{r}{\equiv} 0.$$

Theorem 2.2. *Let R be a homogeneous algebra, I an homogeneous R-ideal, and let us suppose that, locally in true codimension r, I is licci purely of true codimension g and satisfies G_{r+1}. Given the Hilbert series of I^p/I^{p+1} for $0 \leqslant p \leqslant \left[\frac{r-g}{2}\right]$, up to r-equivalence, the Hilbert series of I^p/I^{p+1} can be computed for all p, up to r-equivalence, by the formulas above.*

Proof. Let us choose some sufficiently big integers $d_1, \ldots, d_{r+1}$ (one can treat them as unknowns, or take all of them equal to some fixed number or unknown d). Let us put $q = r + 1 - g$.

First, using $(1)_p$ for $0 \leqslant p \leqslant \left[\frac{q-1}{2}\right]$, we get the Hilbert series of H_p, for p in the same range, up to true codimension $> r$ terms.

Then, from $(2)_p$ for the same p's, we get the series of $H_q, \ldots, H_{q-\left[\frac{q-1}{2}\right]}$. Therefore, we get the series of all the H_p's (up to r-equivalence) if q is odd ; and all of them but one, namely $H_{q/2}$, if q is even. In case q is even, we get the series of $H_{q/2}$ using (3).

Therefore we know the series of all the modules H_p in every case, and can use $(1)_p$ to get the ones of I^p/I^{p+1} for any p.

Notice that the d_i's appear in every formula, but should disappear at the end ! ☐

Example 2.3. *If $X \subseteq \mathbb{P}^n$ is an equidimensionnal locally complete intersection scheme, and $\mathcal{I}_X$ the corresponding ideal sheaf, the Hilbert polynomials of the sheaves $\mathcal{I}_X^p/\mathcal{I}_X^{p+1}$ are all determined by the ones for $0 \leqslant p \leqslant \left[\frac{\dim X}{2}\right]$.*

Proof. It is the case where $r = n$ and $g = n - \dim X$. ☐

Example 2.4. If $X \subseteq \mathbb{P}^n$ is an equidimensionnal locally complete intersection threefold, and $\mathcal{I}_X$ the corresponding ideal sheaf, the Hilbert polynomial of the sheaves $\mathcal{I}_X^p/\mathcal{I}_X^{p+1}$ are all determined by the Hilbert polynomials of X and the one of the conormal bundle $\mathcal{I}_X/\mathcal{I}_X^2$. Moreover the coefficients of terms in degrees 3 and 2 in the Hilbert polynomial of the conormal bundle are determined by those of the Hilbert polynomial for X.

Informally, we have the following picture for the determination of the highest $r-g+1$ coefficients of the Hilbert polynomial of the powers of an ideal I of codimension g that is locally complete intersection up to codimension r :

HP coef.		·	·	g	·	·	·	·	r	·	·	
R/I	$\cdots$	0	0	■	■	■	■	■	■	?	?	$\cdots$
I/I^2	$\cdots$	0	0	□	□	■	■	■	■	?	?	$\cdots$
I^2/I^3	$\cdots$	0	0	□	□	□	□	■	■	?	?	$\cdots$
I^3/I^4	$\cdots$	0	0	□	□	□	□	□	□	?	?	$\cdots$
I^4/I^5	$\cdots$	0	0	□	□	□	□	□	□	?	?	$\cdots$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

■ : needed as input.
□ : may be computed from the others.
? : not concerned.

2.2 Hilbert polynomials of powers of an ideal

We will treat the example of an equidimensional locally complete intersection threefold in projective n-space.

Let us abreviate $e_i(p) = e_i(I^p/I^{p+1})$. Theorem 2.2 asserts that all the coefficients $e_i(p)$ are determined by six of them.

With the help of a computer algebra system, one gets the following formulas.

Formulas 2.5

$$
\begin{aligned}
e_0(p) &= \binom{g+p-1}{p} e_0(0), \\
e_1(p) &= g\binom{g+p-1}{p-1} e_0(0) + \frac{(g+2p)}{(g+p)}\binom{g+p}{p} e_1(0), \\
e_2(p) &= \frac{g(g+1)}{2}\binom{g+p-1}{p-2} e_0(0) + (g+1)\binom{g+p-2}{p-2} e_1(0) \\
&\quad - \frac{(p-1)g}{(g+p)}\binom{g+p+1}{p} e_2(0) + \binom{g+p}{p-1} e_2(1), \\
e_3(p) &= \frac{g(g+1)(g+2)}{6}\binom{g+p-1}{p-3} e_0(0) - \frac{(g+1)(g+2)}{2}\binom{g+p-1}{p-2} e_1(0) \\
&\quad - \frac{p(p-1)(g+2)}{(g+p)}\binom{g+p+1}{p-2} e_2(0) + (g+2)\binom{g+p}{p-2} e_2(1) \\
&\quad + \frac{(p-1)(g+2p)}{(g+p)}\binom{g+p+1}{p} e_3(0) + \frac{(g+2p)}{(g+2)}\binom{g+p}{p-1} e_3(1).
\end{aligned}
$$

Notice that these formulas remains valid for the case of any scheme $X \subseteq \mathbb{P}^n$ that is locally a complete intersection in dimension $\geqslant \dim X - 3$.

As a first guess, one may hope that, at least with some strong hypotheses on X, the Hilbert polynomial of the powers are determined by the Hilbert polynomial of X. This is even not true for complete intersections, due to the following computation.

Suppose that X is a global complete intersection of codimension g and denote by $\sigma_1, \dots, \sigma_g$ the symmetric functions on the degrees of the defining equations of X. Setting

$$\alpha_1 = \sigma_1 - g, \quad \alpha_2 = \sigma_1^2 - 2\sigma_2 - g, \quad \alpha_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 - g,$$

one gets,

$$\begin{aligned} e_0(0) &= \sigma_g, \\ e_1(0) &= \frac{\sigma_g}{2}\alpha_1, \\ e_2(0) &= \frac{\sigma_g}{24}(3\alpha_1^2 - 6\alpha_1 + \alpha_2), \\ e_3(0) &= \frac{\sigma_g}{48}(\alpha_1^3 - 6\alpha_1^2 + 8\alpha_1 + \alpha_1\alpha_2 - 2\alpha_2). \end{aligned}$$

Notice that these formulas imply that $e_3(0)$, the fourth coefficient of the Hilbert polynomial, is a rational function of the first three :

Remark 2.6 If e_i denotes the i-th coefficient of the Hilbert polynomial of a global complete intersection of dimension at least 3 in a projective space, then

$$e_3 = e_2 - \frac{e_1 e_2}{e_0} + \frac{e_1}{6} - \frac{e_1^2}{2e_0} + \frac{e_1^3}{3e_0^2}.$$

Now, using the expansion

$$t^{d_1} + \cdots + t^{d_g} = g + (g + \alpha_1)(t-1) + (\alpha_2 - \alpha_1)\frac{(t-1)^2}{2} + (\alpha_3 - 3\alpha_2 + 2\alpha_1)\frac{(t-1)^3}{6} + \cdots,$$

one can compute the coefficients $e_i(1)$ for $1 \leqslant i \leqslant 3$. The only place where α_3 appears is in $e_3(1) = \frac{\sigma_g}{6}\alpha_3 + \cdots$.

If one chooses two collections of degrees such that the first, second and 4-th symmetric functions are equal but the third one differs, one gets an example of two complete intersections of dimesion three in $\mathbf{P}^7$ having the same Hilbert polynomials (but distinct Hilbert functions !) such that the constant term of the Hilbert polynomials of their conormal bundle are distinct. Such examples were given to us by Benjamin de Weger, the two “smallest” ones are (1,6,7,22)-(2,2,11,21) and (2,6,7,15)-(3,3,10,14). He also gave an infinite collection of them, and Noam Elkies gave a rational parametrization of all the solutions (after a linear change of coordinates the solutions are parametrized by a quadric in $\mathbf{P}^5$).

2.3 The degree of the residual

Let us suppose that the projective scheme X of dimension D is locally a complete intersection in codimension at most s, and use our formulas and the above computations to derive the degree of the codimension s part of an s-residual intersection.

We will treat the cases where $\delta = s - g$ is less or equal to three. As before σ_i stands for the i-th symmetric function on $d_1, \dots, d_s$. We will also set, to simplify some formulas, $e'_i(p) = \sum_{j=0}^{p} e_i(j)$, which is the i-th coefficient of the Hilbert polynomial R/I_X^p.

- If $\delta = 0$, $e_0^D(R/\mathfrak{R}) = \sigma_s - e_0(0)$ (Bézout).
- If $\delta = 1$, $e_0^{D-1}(R/\mathfrak{R}) = \sigma_s - (\sigma_1 - g)e_0(0) + 2e_1(0)$, as proved by Stückrad in [18].
- If $\delta = 2$, using the Taylor expansion of $\sigma_1(t^{d_1}, \dots, t^{d_s})$,

$$\sigma_1(t^{d_1}, \dots, t^{d_s}) = s + \sigma_1(t-1) + (\sigma_1^2 - \sigma_1 - 2\sigma_2)\frac{(t-1)^2}{2} \\ + (\sigma_1^3 - 3\sigma_1^2 + 2\sigma_1 - 3\sigma_2(\sigma_1 - 2) + 3\sigma_3)\frac{(t-1)^3}{6} + \cdots,$$

one recovers the formula given by Huneke and Martin in [16],

$$e_0^{D-2}(R/\mathfrak{R}) = \sigma_s - \left(\sigma_3 - g\sigma_2 + \binom{g+1}{1}\right) e_0(0) + (2\sigma_1 - (g+1))e_1(0) + (g+1)e_2(0) - e'_2(1).$$

- If $\delta = 3$, using the following Taylor expansion,

$$\sigma_2(t^{d_1}, \dots, t^{d_s}) = \frac{s(s-1)}{2} + (s-1)\sigma_1(t-1) + ((s-1)(\sigma_1^2 - \sigma_1 - 2\sigma_2) + 2\sigma_2)\frac{(t-1)^2}{2} \\ + ((s-1)(\sigma_1^3 - 3\sigma_1^2 + 2\sigma_1) - 3(s-2)\sigma_2(\sigma_1 - 2) + 3(s-4)\sigma_3)\frac{(t-1)^3}{6} + \cdots$$

one gets from Theorem 1.4,

$$e_0^{D-3}(R/\mathfrak{R}) = \sigma_s - \left(\sigma_3 - g\sigma_2 + \binom{g+1}{2}\sigma_1 - \binom{g+2}{3}\right) e_0(0) - (2\sigma_2 - (g+1)\sigma_1)e_1(0) \\ + ((g+1)\sigma_1 - (g+2)(g+3))e_2(0) - (\sigma_1 - (g+2))e'_2(1) + 2(g+1)e_3(0) + 2e'_3(1).$$

3. Applications to secant varieties

Theorem 3.1. *Let k be a perfect field, $X \subset \mathbf{P}_k^N$ an eqidimensional subscheme of dimension two with at most isolated licci Gorenstein singularities, A its homogeneous coordinate ring, $\omega := \omega_A$ the canonical module, and $\Omega := \Omega_{A/k}$ the module of differentials.*

(a) *One has*

$$e_0(A)^2 + 14e_0(A) - 16e_1(A) + 4e_2(A) \geqslant e_2(\omega \otimes_A \omega) + e_2(\Omega).$$

(b) *In case the singularities of X have embedding codimension at most two, then equality holds in* (a) *if and only if the secant variety of X is deficient, i.e.*

$$\dim \operatorname{Sec}(X) < 5.$$

Proof. We may assume that k is infinite. We define the ring R and the R-ideal I via the exact sequence

$$0 \longrightarrow I \longrightarrow R := A \otimes_k A \overset{\text{mult}}{\longrightarrow} A \longrightarrow 0.$$

Recall that $\Omega \simeq I/I^2$. The ring R is a standard graded k-algebra of dimension 6 with $\omega_R = \omega \otimes_k \omega$ and $\operatorname{codim} NG(R) > 2$. The ideal I has height 3, and is generated by linear forms. Moreover, I satisfies G_5 and, in the setting of (b), even G_6. Indeed, for any $\mathfrak{p} \in V(I)$, one has $\mu(I_\mathfrak{p}) = \mu(\Omega_\mathfrak{p}) \leqslant \operatorname{ecodim}(A_\mathfrak{p}) + \dim A \leqslant \dim R_\mathfrak{p}$ if $\dim R_\mathfrak{p} \leqslant 4$ or, in the setting of (b), $\dim R_\mathfrak{p} \leqslant 5$. In addition, for every $\mathfrak{p} \in V(I)$ with $\dim R_\mathfrak{p} \leqslant 5$, we have $\operatorname{depth} (I/I^2)_\mathfrak{p} = \operatorname{depth} \Omega_\mathfrak{p} \geqslant \dim A_\mathfrak{p} - 1$. To see this, we write $A_\mathfrak{p} \simeq S/\mathfrak{B}$ with S a regular local ring and $\mathfrak{B}$ a licci Gorenstein ideal. This is possible because $A_\mathfrak{p}$ is licci and Gorenstein. By [4, 6.2.11 and 6.2.12] the module $\mathfrak{B}/\mathfrak{B}^2$ is Cohen-Macaulay. Thus the natural complex

$$0 \to \mathfrak{B}/\mathfrak{B}^2 \to \Omega_{S/k} \otimes_S A_\mathfrak{p} \simeq \oplus A_\mathfrak{p} \to \Omega_{A_\mathfrak{p}/k} \simeq \Omega_\mathfrak{p} \to 0$$

is exact and shows that $\operatorname{depth} \Omega_\mathfrak{p} \geqslant \dim A_\mathfrak{p} - 1$.

Now let $\mathfrak{A}$ be an R-ideal generated by 5 general linear forms in I. Notice that the five general linear forms in I that generate $\mathfrak{A}$ are a weak R-regular sequence off $V(I)$, hence off $V(\mathfrak{A})$. Since $\operatorname{codim} NG(R) > 0$ and $\operatorname{codim} NG(R) \cap V(I) > 5$ it follows that $\operatorname{codim} NG(R) \cap V(\mathfrak{A}) > 5$. By [2, 1.4] one has $\operatorname{ht}(\mathfrak{A} : I) \geqslant 5$ as I satisfies G_5, and $\operatorname{ht}(I + (\mathfrak{A} : I)) \geqslant 6$ in (b) as I is G_6. Thus in the setting of (b), [19] shows that $\operatorname{ht}(\mathfrak{A} : I) \geqslant 6$ if and only if the analytic spread $\ell(I)$ is at most 5. On the other hand $\dim \operatorname{Sec}(X) = \ell(I) - 1$ according to [17]. Hence $\dim \operatorname{Sec}(X) < 5$ if and only if $e_0^1(I/\mathfrak{A}) = 0$.

We now apply Theorem 1.9(a) with $r = s = 5$ and $g = 3$. The theorem yields

(1) $\quad e_0^1(I/\mathfrak{A}) = e_0(R) - e_2(A) - \sum_{j=1}^{2} \sum_{k=0}^{2} (-1)^{j+k} \binom{5}{j+3} \binom{6+j-k}{2-k} e_k(\omega_R / I^j \omega_R).$

Thus the present theorem follows once we have shown that the right hand side of (1) equals

(2) $\quad e_0(A)^2 + 14 e_0(A) - 16 e_1(A) + 4 e_2(A) - e_2(\omega \otimes_A \omega) - e_2(\Omega).$

From [5, IX 2.1] one obtains the isomorphisms of R-modules

$$\omega_R / I\omega_R \cong \omega_R \otimes_R R/I \cong (\omega \otimes_k \omega) \otimes_{A \otimes_k A} A \cong \omega \otimes_A (\omega \otimes_A A) \cong \omega \otimes_A \omega$$

and, since $\operatorname{codim} NG(R) \cap V(I) > 5$,

$$I\omega_R / I^2 \omega_R \underset{5}{\cong} \omega_R \otimes_R I/I^2 \cong (\omega \otimes_k \omega) \otimes_{A \otimes_k A} \Omega \cong \omega \otimes_A (\omega \otimes_A \Omega) \cong (\omega \otimes_A \omega) \otimes_A \Omega.$$

Therefore the right hand side of (1) becomes

(3) $\quad e_0(A)^2 - 7 e_0(A) - e_2(A) - 23 e_1(\omega^{\otimes 2}) + 4 e_2(\omega^{\otimes 2}) + 7 e_1(\omega^{\otimes 2} \otimes \Omega) - e_2(\omega^{\otimes 2} \otimes \Omega).$

Here and in what follows tensor products are taken over the ring A.

We are now going to express the Hilbert coefficients $e_1(\omega^{\otimes 2})$, $e_1(\omega^{\otimes 2} \otimes \Omega)$ and $e_2(\omega^{\otimes 2} \otimes \Omega)$ in terms of the Hilbert coefficients of A, $e_2(\omega^{\otimes 2})$ and $e_2(\Omega)$. First notice that for any finitely generated graded A-module M,

$$(4) \quad e_i(M(-1)) = e_i(M) + e_{i-1}(M).$$

Hence, by Lemma 1.7 and Remark 1.8,

$$(5) \quad e_1(\omega) = 3e_0(A) - e_1(A) \quad \text{and} \quad e_2(\omega) = 3e_0(A) - 2e_1(A) + e_2(A).$$

Since ω is free of rank 1 locally in codimension 1, there is a complex of graded A-modules

$$(6) \quad 0 \longrightarrow Z \longrightarrow A(-a)^2 \longrightarrow \omega \longrightarrow 0$$

for some $a \gg 0$, that is exact in codimension 1. It induces complexes

$$(7) \quad 0 \longrightarrow Z \otimes A(-ja+a)^j \longrightarrow A(-ja)^{j+1} \longrightarrow \mathrm{Sym}_j(\omega) \underset{1}{\cong} \omega^{\otimes j} \longrightarrow 0$$

that are likewise exact locally in codimension 1. Now (6) yields $e_1(Z) = 2e_1(A(-a)) - e_1(\omega)$ and then (4), (5) and (7) show that for every $j \geqslant 0$,

$$(8) \quad e_1(\omega^{\otimes j}) = 3je_0(A) - (2j-1)e_1(A).$$

Now, we treat the first Hilbert coefficient of $\omega^{\otimes 2} \otimes \Omega$. Since ω is free of rank 1 locally in codimension 2, we also have $\omega^* \xrightarrow[2]{\sim} \mathrm{Hom}_A(\omega^{\otimes 2}, \omega)$, which by Lemma 1.7 and Remark 1.8 gives

$$(9) \quad e_1(\omega^*) = 3e_0(\omega^{\otimes 2}) - e_1(\omega^{\otimes 2}) \text{ and } e_2(\omega^*) = 3e_0(\omega^{\otimes 2}) - 2e_1(\omega^{\otimes 2}) + e_2(\omega^{\otimes 2}).$$

Furthermore,

$$(10) \quad \omega^{\otimes 2} \otimes \omega^* \xrightarrow[2]{\sim} \mathrm{Hom}_A(\omega, \omega^{\otimes 2}) \xleftarrow[2]{\sim} \omega.$$

As Ω is free of rank 3 locally in codimension 1, there is an exact sequence of graded A-modules

$$0 \longrightarrow A(-1)^2 \longrightarrow \Omega \longrightarrow C \longrightarrow 0$$

where C is free of rank 1 locally in codimension 1. Thus

$$C \otimes \bigwedge^2 (A(-1)^2) \xrightarrow[1]{\sim} \bigwedge^3 \Omega \xrightarrow[1]{\sim} \omega,$$

which gives a complex

$$0 \longrightarrow A(-1)^2 \longrightarrow \Omega \longrightarrow \omega(2) \longrightarrow 0$$

that is exact in codimension 1. Tensoring with $\omega^{\otimes 2}$ we obtain

$$0 \longrightarrow \omega^{\otimes 2}(-1)^2 \longrightarrow \omega^{\otimes 2} \otimes \Omega \longrightarrow \omega^{\otimes 3}(2) \longrightarrow 0.$$

Since this complex is exact in codimension 1, (4) and (8) imply that

(13) $\quad e_1(\omega^{\otimes 2} \otimes \Omega) = 21e_0(A) - 11e_1(A).$

Next, we turn to the second Hilbert coefficient of $\omega^{\otimes 2} \otimes \Omega$. Write $—^* := \mathrm{Hom}_A(—, A)$ and $e := N - 2$. Increasing N if needed, we may assume $e \geqslant 2$. We define a graded A-module E via the exact sequence

(14) $\quad 0 \longrightarrow E \longrightarrow A(-1)^{e+3} \longrightarrow \Omega \longrightarrow 0.$

Notice that E has rank e, and is free locally in codimension 1 and Cohen-Macaulay locally in codimension 2. Futhermore (14) gives

(15) $\quad (\bigwedge^e E)^{**} \underset{2}{\xrightarrow{\sim}} (\bigwedge^3 \Omega)^*(-e-3) \underset{2}{\xleftarrow{\sim}} \omega^*(-e-3).$

As E^* is free locally in codimension 1 and $\operatorname{rk} E^* - 1 \geqslant 1$, there exists a homogeneous element $f \in E^*$ of degree $c \gg 0$ whose order ideal $(E^*)^*(f)$ has height at least 2 (see [8]). However, the ideals $E^{**}(f)$ and $J := f(E)$ coincide locally in codimension 1 since E is reflexive locally in codimension 1. Hence $\operatorname{ht} J = \operatorname{ht} E^{**}(f) \geqslant 2$. The map f induces an exact sequence of graded A-modules

$$0 \longrightarrow E_{e-1} \longrightarrow E_e := E \longrightarrow J_e(c_e) := J(c) \longrightarrow 0.$$

Repeating this procedure, if needed, we obtain a filtration

(16) $\quad E_1 \subset E_2 \subset \cdots \subset E_e \quad \text{with} \quad E_i/E_{i-1} \cong J_i(c_i),$

where J_i are homogeneous A-ideals of height at least 2. Thus E_i has rank i, is free in codimension 1 and Cohen-Macaulay in codimension 2, and

$$(\bigwedge^{i-1} E_{i-1})^{**}(c_i) \underset{2}{\xleftarrow{\sim}} (\bigwedge^{i-1} E_{i-1})^{**} \otimes (J_i(c_i))^{**} \underset{2}{\xrightarrow{\sim}} (\bigwedge^{i} E_i)^{**}.$$

Since E_1 is reflexive locally in codimension 2, it follows that

$$E_1 \underset{2}{\xrightarrow{\sim}} E_1^{**} \underset{2}{\cong} (\bigwedge^{e} E)^{**}(-\sum_{i=2}^{e} c_i),$$

which together with (15) implies

(17) $\quad E_1 \underset{2}{\cong} \omega^*(-e-3-\sum_{i=2}^e c_i).$

The exact sequence

$$0 \longrightarrow J_i(c_i) \longrightarrow A(c_i) \longrightarrow (A/J_i)(c_i) \longrightarrow 0$$

yields a complex

$$0 \longrightarrow \omega^{\otimes 2} \otimes J_i(c_i) \longrightarrow \omega^{\otimes 2}(c_i) \longrightarrow (\omega^{\otimes 2}/\omega^{\otimes 2} J_i)(c_i) \longrightarrow 0$$

that is exact in codimension 2. Since $\operatorname{ht} J_i \geqslant 2$ and $\omega^{\otimes 2}$ is free of rank 1 locally in codimension 2, it follows that $e_0^1((A/J_i)(c_i)) = e_0^1((\omega^{\otimes 2}/\omega^{\otimes 2}J_i)(c_i))$. We conclude

$$e_2(J_i(c_i)) - e_2(\omega^{\otimes 2} \otimes J_i(c_i)) = e_2(A(c_i)) - e_2(\omega^{\otimes 2}(c_i)). \tag{18}$$

Tensoring (14) and (16) with $\omega^{\otimes 2}$ and using (18) and (17) we obtain

$$\begin{aligned} e_2(\omega^{\otimes 2} \otimes \Omega) - e_2(\Omega) &= e_2(\omega^{\otimes 2}(-1)^{e+3}) - e_2(A(-1)^{e+3}) + \sum_{i=2}^{e}(e_2(A(c_i)) - e_2(\omega^{\otimes 2}(c_i))) \\ &\quad + e_2(\omega^*(-e-3-\sum_{i=2}^{e} c_i)) - e_2(\omega^2 \otimes \omega^*(-e-3-\sum_{i=2}^{e} c_i)). \end{aligned} \tag{19}$$

Combining (19) with (4), (8), (9), (10), (5), we deduce

$$e_2(\omega^{\otimes 2} \otimes \Omega) = -12e_0(A) + 8e_1(A) - 5e_2(A) + 5e_2(\omega^{\otimes 2}) + e_2(\Omega). \tag{20}$$

Substituting (8), (13), (20) into (3), we conclude that (3) and (2) coincide. □

Corollary 3.2. Let k be a perfect field, $X \subset \mathbf{P}_k^4$ an equidimensional subscheme of dimension two with at most isolated Gorenstein singularities and A its homogeneous coordinate ring. One has

$$e_0(A)^2 + 14e_0(A) - 16e_1(A) + 4e_2(A) = e_2(\omega_A \otimes_A \omega_A) + e_2(\Omega_{A/k}).$$

Remark. The inequality in Theorem 3.1 can be replaced by

$$e_0(A)^2 + 5e_0(A) - 10e_1(A) + 4e_2(A) \geqslant e_2(\omega_A^*) + e_2(\Omega_{A/k}).$$

Proof. Use equalities (9) and (8) in the proof of Theorem 3.1. □

Corollary 3.3. Let k be a field, $X \subset \mathbf{P}_k^N$ an equidimensional smooth subscheme of dimension two, H the class of the hyperplane section, K the canonical divisor, and c_2 the second Chern class of the cotangent bundle of X. One has

$$(H^2)^2 \geqslant 10H^2 + 5HK + K^2 - c_2,$$

and equality holds if and only if $\dim \operatorname{Sec}(X) < 5$.

Proof. The Riemannn-Roch theorem in dimension two gives

$$\chi(X,E) = \frac{1}{2}[c_1(E)^2 - 2c_2(E) - c_1(E)K_X] + \operatorname{rk} E\,\chi(X,\mathcal{O}_X).$$

If D is a divisor this equality specializes to

$$\begin{aligned} \chi(D+nH) &= \frac{1}{2}H^2n^2 + (DH - \frac{1}{2}KH)n + \frac{1}{2}(D^2 - KD) + \chi(X,\mathcal{O}_X) \\ &= H^2\binom{n+2}{2} - \frac{1}{2}(3H^2 + KH - 2DH)\binom{n+1}{1} \\ &\quad + \frac{1}{2}(H^2 + KH - 2DH - KD + D^2) + \chi(X,\mathcal{O}_X). \end{aligned}$$

For a rank two vector bundle E the formula reads

$$\begin{aligned}\chi(E+nH) &= H^2n^2 + (c_1(E)H - KH)n + \frac{1}{2}(c_1(E)^2 - Kc_1(E)) - c_2(E) + 2\chi(X,\mathcal{O}_X)\\ &= 2H^2\binom{n+2}{2} - (3H^2 + KH - c_1(E)H)\binom{n+1}{1}\\ &\quad + H^2 + KH - c_1(E)H - \frac{1}{2}Kc_1(E) + \frac{1}{2}c_1(E)^2 - c_2(E) + 2\chi(X,\mathcal{O}_X).\end{aligned}$$

Taking $D = 0$ we obtain

$$\begin{aligned}e_0(\mathcal{O}_X) &= H^2\\ e_1(\mathcal{O}_X) &= \frac{3}{2}H^2 + \frac{1}{2}KH\\ e_2(\mathcal{O}_X) &= \frac{1}{2}H^2 + \frac{1}{2}KH + \chi(X,\mathcal{O}_X),\end{aligned}$$

and for $D = 2K$

$$\begin{aligned}e_0(\omega_X^{\otimes 2}) &= H^2\\ e_1(\omega_X^{\otimes 2}) &= \frac{3}{2}H^2 - \frac{3}{2}KH\\ e_2(\omega_X^{\otimes 2}) &= \frac{1}{2}H^2 - \frac{3}{2}KH + K^2 + \chi(X,\mathcal{O}_X).\end{aligned}$$

Finally, taking $E = \Omega_X$, the cotangent sheaf of X, and using the fact that $c_1(\Omega_X) = K$ we deduce

$$\begin{aligned}e_0(\Omega_X) &= 2H^2\\ e_1(\Omega_X) &= 3H^2\\ e_2(\Omega_X) &= H^2 - c_2(\Omega_X) + 2\chi(X,\mathcal{O}_X).\end{aligned}$$

Now the assertion of the remark follows from the Theorem since $e_i(A) = e_i(\mathcal{O}_X)$, $e_i(\omega) = e_i(\omega_X)$ and $e_i(\Omega) = e_i(\Omega_X) + e_i(\mathcal{O}_X)$. □

Theorem 3.4. Let k be a field, $X \subset \mathbf{P}_k^N$ an equidimensional smooth subscheme of dimension three, H the class of the hyperplane section, K the canonical divisor, and c_2 and c_3 the second and third Chern class of the cotangent bundle of X. One has

$$(H^3)^2 \leqslant 35H^3 - 11H^2K - 9K^2H + c_2H - K^3 - \frac{1}{12}Kc_2 + \frac{1}{2}c_3,$$

and equality holds if and only if $\dim \operatorname{Sec}(X) < 7$.

Proof. We use the notation of Theorem 3.1, taking $\mathfrak{A}$ to be generated by 7 general linear forms in the ideal I of the diagonal. Recall that $\dim \operatorname{Sec}(X) < 7$ if and only if $\operatorname{ht}(\mathfrak{A} : I) \geqslant 8$.

We will apply Theorem 1.9 (a) with $r = s = 7$ and $g = 4$. Because X is smooth we have

$$\begin{aligned}I\omega_R/I^2\omega_R &\underset{7}{\cong} \omega^2 \otimes \Omega\\ I^2\omega_R/I^3\omega_R &\underset{7}{\cong} \omega^2 \otimes S_2\Omega.\end{aligned}$$

As X is smooth, we can apply the Riemann-Roch formula as in the proof of Corollary 3.3, to derive the following formulas, which express the Hilbert coefficients used in Theorem 1.9(a) in terms of the numbers that appear in our desired formula:

$$
\begin{aligned}
e_0(A) &= H^3\\
e_1(A) &= 2H^3 - \tfrac{3}{2}KH^2\\
e_2(A) &= \tfrac{1}{12}(14H^3 + 9KH^2 + K^2H + c_2H)\\
e_3(A) &= \tfrac{1}{24}(4H^3 + 6KH^2 + 2K^2H + 2c_2H + Kc_2)
\end{aligned}
$$

$$
\begin{aligned}
e_0(\Omega) &= 4H^3\\
e_1(\Omega) &= 8H^3 + KH^2\\
e_2(\Omega) &= \tfrac{1}{6}(28H^3 + 9KH^2 + 2K^2H - 4c_2H)\\
e_3(\Omega) &= \tfrac{1}{12}(8H^3 + 6KH^2 + 4K^2H - 8c_2H + Kc_2 - 6c_3)
\end{aligned}
$$

$$
\begin{aligned}
e_0(\omega^{\otimes 2}) &= H^3\\
e_1(\omega^{\otimes 2}) &= 2H^3 - \tfrac{3}{2}KH^2\\
e_2(\omega^{\otimes 2}) &= \tfrac{1}{12}(14H^3 - 27KH^2 + 13K^2H + c_2H)\\
e_3(\omega^{\otimes 2}) &= \tfrac{1}{24}(4H^3 - 18KH^2 + 26K^2H + 2c_2H - 12K^3 - 3Kc_2)
\end{aligned}
$$

$$
\begin{aligned}
e_0(\omega^{\otimes 2} \otimes \Omega) &= 4H^3\\
e_1(\omega^{\otimes 2} \otimes \Omega) &= 8H^3 - 7KH^2\\
e_2(\omega^{\otimes 2} \otimes \Omega) &= \tfrac{1}{6}(28H^3 - 63KH^2 + 38K^2H - 4c_2H)\\
e_3(\omega^{\otimes 2} \otimes \Omega) &= \tfrac{1}{12}(8H^3 - 42KH^2 + 76K^2H - 8c_2H - 48K^3 + 17Kc_2 - 6c_3)
\end{aligned}
$$

$$
\begin{aligned}
e_0(\omega^{\otimes 2} \otimes S_2\Omega) &= 10H^3\\
e_1(\omega^{\otimes 2} \otimes S_2\Omega) &= 20H^3 - 19KH^2\\
e_2(\omega^{\otimes 2} \otimes S_2\Omega) &= \tfrac{1}{6}(70H^3 - 171KH^2 + 119K^2H - 25c_2H)\\
e_3(\omega^{\otimes 2} \otimes S_2\Omega) &= \tfrac{1}{12}(20H^3 - 114KH^2 + 238K^2H - 50c_2H - 186K^3 + 125Kc_2 - 42c_3) \qquad \square
\end{aligned}
$$

Corollary 3.5. *Let k be a perfect field, $X \subset \mathbf{P}^N_k$ an equidimensional smooth subscheme of dimension three, A its homogeneous coordinate ring, $\omega := \omega_A$ the canonical module, and $\Omega := \Omega_{A/k}$ the module of differentials. One has*

$$e_0(A)^2 + 391e_0(A) - 246e_1(A) + 66e_2(A) + 50e_3(A) \geqslant 18e_2(\omega \otimes_A \omega) - 2e_3(\Omega) - 2e_3(\omega \otimes_A \omega)$$

and equality holds if and only if $\dim \operatorname{Sec}(X) < 7$.

Proof. Use the formulas given in Theorem 3.4 to express all the necessary Hilbert coefficients in terms of $e_0, \ldots, e_3$. $\square$

Remark 3.6. If H is the class of the hyperplane section, K the canonical divisor, $D = H^3$ the degree of X, the above inequality is equivalent to:

$$D^2 \geqslant 7(5D + 3KH^2 + K^2H - c_2(\Omega_X)H) - 2c_2(\Omega_X)K + K^3 + c_3(\Omega_X).$$

In other words, if C and S are respectively a curve and a surface obtained by taking general linear sections of X, the formula reads

$$D^2 \geqslant 7(5D + 3\chi_C + 12\chi(\mathcal{O}_S) - 2\chi_S) - 48\chi(\mathcal{O}_X) + K_X^3 + \chi_X.$$

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